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The optimal dimensions of circular fins with variable profile and temperature-dependent thermal conductivity

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Abstract—The optimal dimensions of circular fins with variable profile, and temperature-dependent thermal conductivity are obtained. A profile of the form $y = (w/2)[1+(r_0/r)^n]$ is studied, while the thermal conductivity considered is of the form $k = k_0[1+\varepsilon(T-T_\infty)/T_0]$. The results have been expressed in terms of suitable dimensionless parameters. A correlation for the optimal dimensions of a constant and variable profile fins is presented in terms of reduced heat-transfer rate. It is found that a parabolic circular fin with $n = 2$ gives an optimum performance. For example, an increase in heat-transfer rate (as compared to constant thickness fin) by about 20% for the optimal fin profile is observed. The effect of thermal conductivity on the optimal dimensions is negligible for the variable profile fin. It is also observed that in general, the optimal fin length is greater for the optimal fin profile. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

A large number of engineering problems require high-performance heat transfer components with progressively smaller weights, volumes, costs or accommodating shapes. Circular fins are one of such heat exchanging devices that are used extensively to increase heat-transfer rates, particularly in compact heat exchangers for the aerospace industry. The design and optimization of these fins is generally based on two approaches [1–4]. In the first approach, a profile is selected and based on this, optimal dimensions of the fin are determined. The second approach is to maximize the heat dissipation for a given volume of the fin. For purely conductive and convective fins, the criterion for the optimal fin problem was first proposed by Schmidt [5]. He argued intuitively that there not only exists an optimum fin size when the fin profile is specified, but also an optimum fin profile that maximizes the heat-transfer rate. Schmidt's predictions were confirmed later by Duffin [6], who used a variational calculus approach to the fin problem.

Fin analysis considering temperature-dependent thermal conductivity and internal heat generation was presented by Hung and Appl [7] and Aziz [8]. However, the optimum shape for straight fins and spines with temperature-dependent conductivity was investigated by Jany and Bejan [9]. There are several other studies to obtain optimum fin shapes with different assumptions [10–14]. Razelos and Imre [15] studied the optimal dimensions of circular fins with a trapezoidal profile (constant slope) and variable thermal properties. They showed that for constant thermal conductivity, the optimum base thickness and volume of the fin are inversely proportional to the conductivity of the fin material, while the optimum length and fin effectiveness are independent of the fin-material properties.

In this paper, optimization of circular fins with a general power-law profile and temperature-dependent thermal conductivity is considered. The problem is to maximize the heat-transfer rate for a given fin volume. The optimization variables are the fin length and power of the profile function. It is assumed that the predominant modes of heat transfer are conduction and convection and the effect of radiation is ignored.

MATHEMATICAL FORMULATION

Consider a circular fin of a homogeneous material, with a symmetric profile, attached to a cylindrical surface of radius r_0 with a base temperature T_0, which is measured in excess to the ambient fluid temperature, T_∞. The profile of the fin is $y \equiv y(r)$, the thermal conductivity $k(T)$, and heat-transfer coefficient h. It is assumed that both faces of the fin are exposed to an environment at temperature T_∞. The geometry of the fin is shown schematically in Fig. 1. Assuming the fin to have one-dimensional conduction, the equation for steady-state temperature of the fin can be written as [15–17]

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[k(2\pi r)[2y(r)]\frac{\mathrm{d}T}{\mathrm{d}r}\right]\mathrm{d}r = 2h(2\pi r)\,\mathrm{d}s(T-T_\infty) \quad (1)$$

where

NOMENCLATURE

B_r	dimensionless parameter ($B_r = hr_0/k_0$)
E_f	fin effectiveness ($E_f = q/q_0$)
h	heat-transfer coefficient [$W \cdot m^{-2} \cdot K^{-1}$]
k	thermal conductivity [$W \cdot m^{-1} \cdot K^{-1}$]
L	length of the fin [m]
n	parameter describing the variation of the profile
q	heat-transfer rate [W]
Q	dimensionless heat transfer ($Q = q/4\pi r_0^2 h T_0$)
r	radial coordinate [m]
T	temperature [K]
T_0	base temperature in excess to the ambient temperature [K]
U	dimensionless volume ($U = k_0 V/4\pi r_0^4 h$)
V	volume of the fin [m^3]
w	fin's base half thickness [m]
y	longitudinal coordinate [m].

Greek symbols

β	ratio of radii ($\beta = r_e/r_0$)
ε	parameter describing the variation of thermal conductivity
η	dimensionless fin profile [$\eta = y(r)/w$]
ξ	dimensionless coordinate ($\xi = r/r_0$)
ν	dimensionless parameter [$\nu = r_0(h/wk)^{1/2}$]
θ	dimensionless temperature [$\theta = (T-T_\infty)/T_0$].

Subscripts

al	aluminum
b	base
c	corrected
cu	copper
e	fin tip
0	bore
opt	optimal
∞	ambient.

$$(ds)^2 = (dr)^2 + (dy)^2 \tag{2a}$$

$$ds = dr(1+y'^2)^{1/2} \tag{2b}$$

$$y' = \frac{dy}{dr}. \tag{2c}$$

On substituting equations (2a–2c) into equation (1), we get

$$\frac{d}{dr}\left(kyr\frac{dT}{dr}\right) = hr(T-T_\infty)(1+y'^2)^{1/2}, \tag{3}$$

and the boundary conditions are,

$$T(r_0) = T_b = (T_0 + T_\infty) \tag{4a}$$

$$\left[k\frac{dT}{dr} + h(T-T_\infty)\right]_{r=r_e} = 0, \quad \text{if } y(r_e) = w_e \neq 0 \tag{4b}$$

$$T(r_e) \quad \text{is bounded if } y(r_e) = 0. \tag{4c}$$

The fin-optimization problem is defined as follows: given the volume of the fin

$$V = 4\pi\int_{r_0}^{r_e} y(r) r\, dr, \tag{5}$$

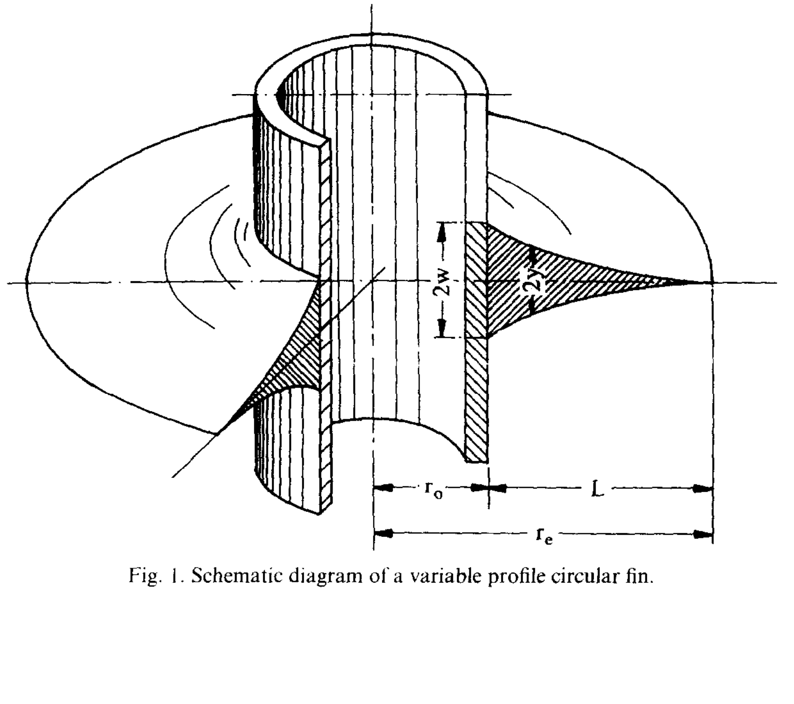

Fig. 1. Schematic diagram of a variable profile circular fin.

we seek the bore half thickness w, the length $L = (r_e - r_0)$, and the power in the profile function n, that maximizes the heat dissipated by the fin. It should be noted that under steady-state conditions, the heat transfer from the fin can be expressed as

$$q_f = -4\pi r_0 w k\left(\frac{dT}{dr}\right)_{r=r_0}, \tag{6}$$

with the constraint that

$$q_f > q_b = 4\pi r_0 w h_b(T_b - T_\infty), \tag{7}$$

where q_b, h_b, T_b refer to the values of the corresponding quantities if no fin were present. We also assume that the thermal conductivity of the fin material is expressed as

$$k = k_0[1 + \varepsilon(T - T_\infty)/T_0], \tag{8}$$

and the fin profile is given by

$$y(r) = \frac{w}{2}\left[1 + \left(\frac{r_0}{r}\right)^n\right]. \tag{9}$$

On integrating equation (5) where $y(r)$ is given by the above equation, we get

$$V = w\left[\pi(r_e^2 - r_0^2) + \frac{2\pi r_0^n}{(2-n)}(r_e^{2-n} - r_0^{2-n})\right]. \tag{10}$$

For the sake of generalization, it is appropriate to nondimensionalize the above equations similar to those of Razelos and Imre [15]. Introducing dimensionless (or reduced) variables

$$\theta = (T - T_\infty)/T_0, \xi = r/r_0, \eta = y(r)/w, \tag{11}$$

and substituting in equation (1), we get

$$\frac{d}{d\xi}\left(k\eta w\xi\frac{d\theta}{d\xi}\right) = h\xi r_0^2\theta\left[1 + \frac{w^2}{r_0^2}\left(\frac{d\eta}{d\xi}\right)^2\right]^{1/2} \quad \text{or} \tag{12}$$

$$\frac{d}{d\xi}\left((1+\varepsilon\theta)\xi\eta\frac{d\theta}{d\xi}\right) = \xi\theta v^2\left[1 + \left(\frac{B_r\eta'}{v^2}\right)^2\right]^{1/2} \tag{13}$$

where

$$v = \left(\frac{h}{wk_0}\right)^{1/2} r_0; \quad B_r = \frac{hr_0}{k_0}; \quad \beta = \frac{r_e}{r_0}. \tag{14}$$

The boundary conditions, given by equation (4), in their nondimensionalized form, are

$$\xi = 1 \quad \theta(1) = 1; \tag{15a}$$

and

$$\frac{d\theta}{d\xi} + \left(\frac{h_e}{h}\right)\frac{B_r\theta}{(1+\varepsilon\theta)} = 0 \quad \text{at } \xi = \beta \quad \text{if } \eta(\beta) \neq 0; \tag{15b}$$

$$\theta(\xi = \beta), \quad \text{is bounded if } \eta(\beta) = 0. \tag{15c}$$

The dimensionless forms of the profile equations are

$$\eta = \frac{1}{2}\left[1 + \left(\frac{1}{\xi}\right)^n\right] \tag{16}$$

and

$$\eta' = -\frac{n}{2}\left(\frac{1}{\xi}\right)^{n+1}, \tag{17}$$

where, v, β and n are parameters to be determined by the optimization process. The dimensionless temperature θ, is obtained by solving the above problem as a function of spatial variable ξ and the parameters β, v, ε, B_r and n. The reduced volume U may be defined as [15]

$$U \equiv \frac{k_0 V}{4\pi r_0^4 h} = \frac{1}{4v^2}\left[(\beta^2 - 1) + \frac{2}{2-n}(\beta^{2-n} - 1)\right], \tag{18}$$

and the functional dependence of the thermal conductivity in dimensionless form becomes

$$K = k/k_0 = (1 + \varepsilon\theta). \tag{19}$$

Now, the problem at hand is in terms of reduced variables. For a given value of U, we can calculate the values of v, β and n which will maximize the reduced heat dissipation defined as [15]

$$Q \equiv \frac{q_f}{4\pi r_0^2 h T_0}, \tag{20}$$

or on using equation (6), we get

$$Q = \frac{-(1+\varepsilon\theta)}{v^2}\left.\frac{d\theta}{d\xi}\right|_{\xi=1}. \tag{21}$$

The optimal values should be such that they satisfy the condition given by equation (7), which can be expressed in dimensionless form as

$$E_f \equiv \frac{q_f}{q_b} = \frac{Qv^2}{B_r}\left(\frac{h}{h_b}\right) > 1, \tag{22}$$

where E_f is defined as the effectiveness of the fin.

SOLUTION METHODOLOGY

Following Razelos and Imre [15], the boundary-value problem given by equations (13)–(15) can be reduced to two first-order ordinary differential equations by defining $X_1 \equiv \theta$; $X_2 \equiv K\eta\xi\theta'$. This gives

$$X_1' = \frac{X_2}{\eta\xi(1+\varepsilon X_1)} \tag{23}$$

$$X_2' = \xi X_1 v^2\left\{1 + \left[\frac{B_r}{v^2}\eta'\right]^2\right\}^{1/2} \tag{24}$$

$$X_1(1) = 1 \quad \text{and} \quad \frac{X_2(\beta)}{\eta\xi} + \left(\frac{h_e}{h}\right) B_r X_1(\beta) = 0. \tag{25}$$

After the above manipulations, the problem is in a form suitable for numerical solution. For optimization and integration, we have used a simulation and optimization package [18]. It uses the subroutine DBVPFD of IMSL library [19] for integration of equations (23)–(25). The integration routine DBVPFD uses a variable order, variable step size finite-difference method with deferred corrections. The optimization procedure uses Feasible Sequential Quadratic Programming FSQP [20]. It is a set of FORTRAN subroutines for the minimization of a nonlinear, smooth objective function subject to both equality and inequality constraints, with simple bounds on the variables.

We emphasize that the objective of this study is to maximize Q, given by equation (21) subject to the constraint (22). It should be noted that the problem is formulated as a nonlinear optimization problem similar to the one presented in the Appendix. Note that the problem at hand has only two parameters (n, r_e) which need to be determined by the optimization process. This will in turn help us calculate w. At any point in the search for an optimal solution, these two parameters are calculated by FSQP. Subsequently, these values are then used in the IMSL routine for integrating equations (23)–(25). Once the integration is performed, the calculated values of Q and the constraints E_f are fed back to the optimizer FSQP, based on which the search proceeds. The flow chart of the method discussed above is presented in Fig. 2. The program that was developed is written in such a way that it maintains the maximum generality. The salient features of the program are:

- include or suppress the effect of convective tip;
- include or suppress the effect of variable thermal conductivity;
- change the profile function to any desired value.

RESULTS AND DISCUSSION

The optimum values of v_{opt}, $U_{opt}^{1/2}$, and β_{opt} are plotted against Q for the constant thickness fin with $\varepsilon = 0$, in Fig. 3. These results are exactly the same as those of ref. [15]. It should be noted that this validation of optimum fin dimensions for a constant thickness annular fin confirms the fact that the optimization technique (FSQP) works for the problem under investigation.

The plots for the optimum values of v_{opt}, $U_{opt}^{1/2}$, β_{opt}, n vs Q for the fin with variable profile of the form $y = (w/2)[1 + (r_0/r)^n]$ are shown in Fig. 4. We emphasize that the plot of β_{opt} vs Q in this figure shows that for a given Q, the optimum dimensions of the fin can be read directly. In addition, Fig. 4 shows a somewhat higher value of reduced fin radius (β_{opt}) compared to the constant thickness fin. It can be seen from the figure that for a given volume of fin, the heat transfer Q for the optimal fin profile increases by about 20% as compared to a constant thickness fin. For example, for the optimal reduced volume, $U_{opt} = 1$ gives the reduced heat transfer $Q = 0.9$ for the constant thickness fin (refer to Fig. 3) compared to $Q = 1.10$ for the optimal fin profile (refer to Fig. 4). This is further demonstrated later by an illustrative example. It is also interesting to note that the value of n stays constant, and is equal to 2 for the variable fin profile. That is, a circular fin of parabolic profile gives an optimum performance.

The variation of the heat transfer Q with respect to the parameter ε was studied by varying $-0.4 \leqslant \varepsilon \leqslant$

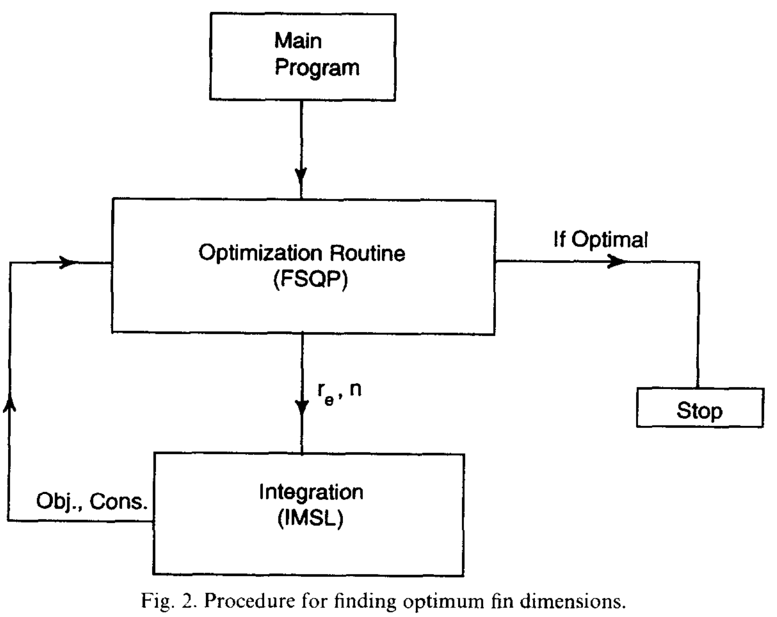

Fig. 2. Procedure for finding optimum fin dimensions.

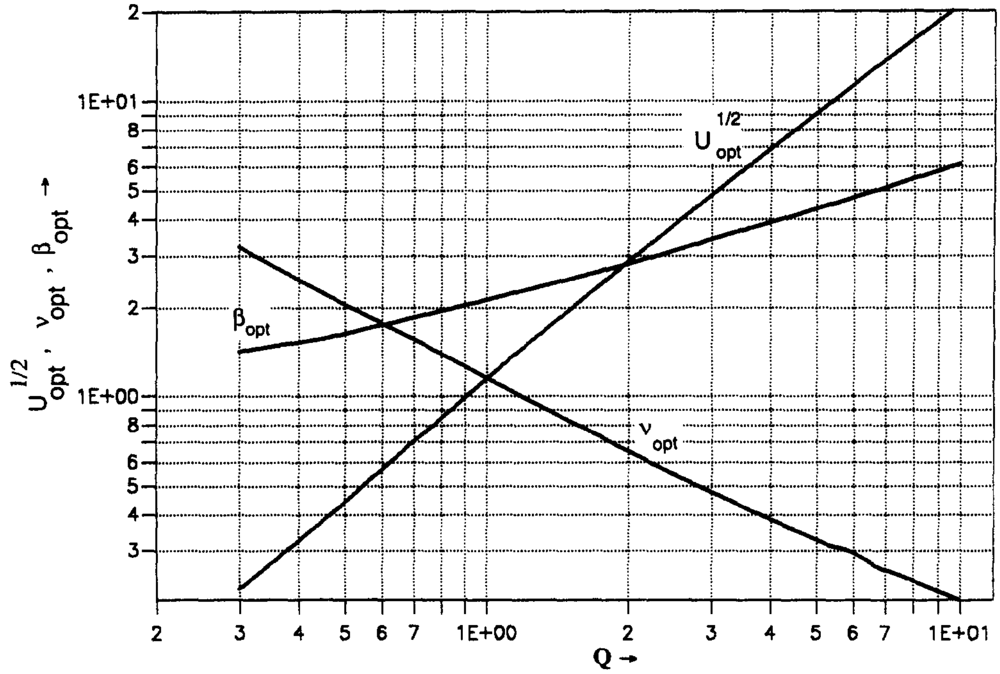

Fig. 3. Optimal reduced dimensions of a straight circular fin vs the reduced heat-transfer rate, with constant thermal conductivity.

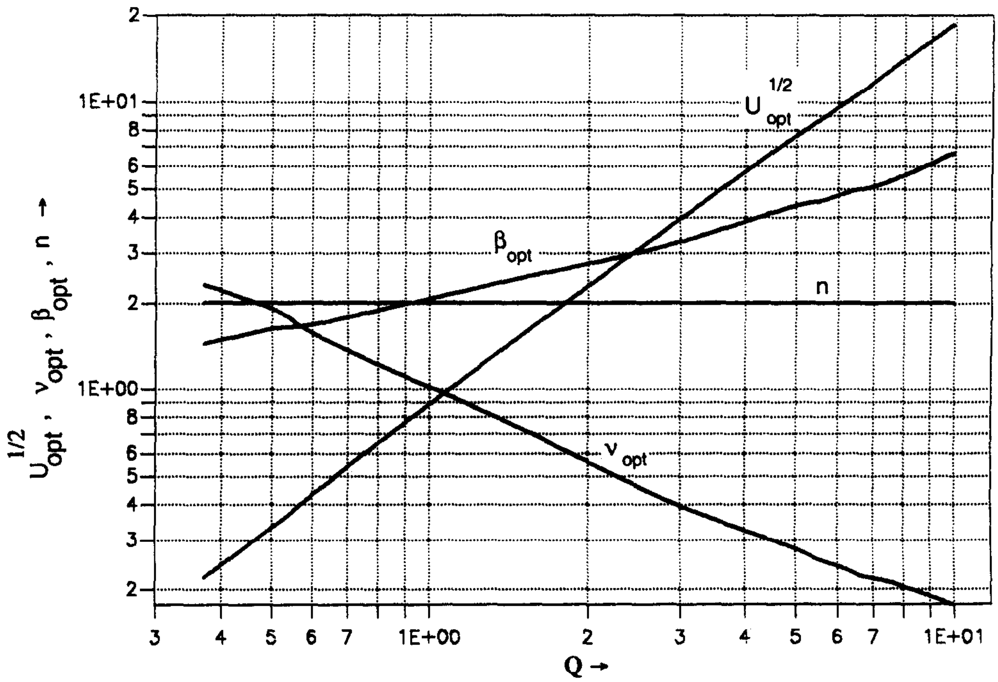

Fig. 4. Optimal reduced dimensions of a variable-profile circular fin vs the reduced heat-transfer rate, with constant thermal conductivity.

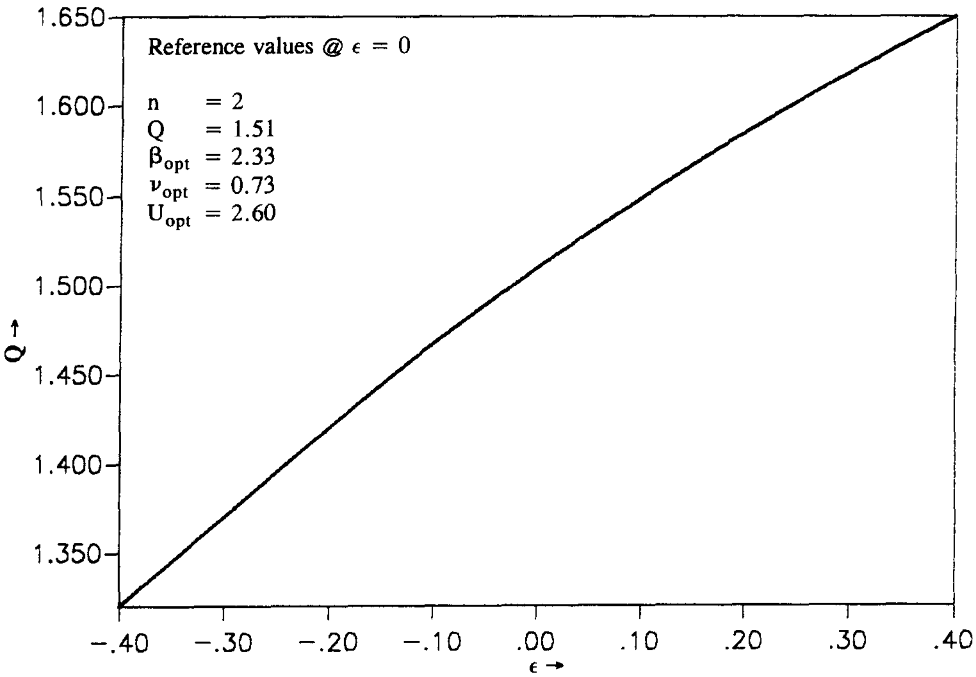

Fig. 5. The reduced heat-transfer rate (Q) vs variable thermal conductivity constant (ε) for an optimum profile fin.

0.4. The results are presented in Fig. 5. It is observed from this figure that Q shows an increasing trend as ε is increased from -0.4 to 0.4. The optimal dimensions are not affected and show no appreciable change. This observation is in agreement with the results for a trapezoidal profile (constant slope) fin presented in ref. [15].

Correlation for the optimal dimensions

It is useful to present the numerical results obtained for the optimal dimensions in the form of regression equations. An attempt was made to find the correlation for the optimal dimensions (v_{opt}, β_{opt}, U_{opt}) by curve fitting these variables as a function of Q. Regression analysis was carried out by using the statistical analysis package STATGRAPHICS [21]. The following functional forms were found to be most suitable.

$$v_{opt} = a_1 + (b_1/Q^{c_1}) \tag{26}$$

$$\beta_{opt} = a_2 + b_2 Q^{c_2}, \tag{27}$$

$$U_{opt} = a_3 Q^{b_3}, \tag{28}$$

where a_1, b_1, c_1, a_2, b_2, c_2, a_3 and b_3 are regression constants determined by regression analysis using the statistical analysis package [21]. The results for these parameters are presented in Table 1.

To incorporate the effect of variable thermal conductivity on the heat-transfer rate, a regression analysis was carried out between ε and Q. The value of Q at $\varepsilon = 0$ is considered as a reference value (Q_0) and the corrected value of reduced heat-transfer rate (Q_c) is modelled as

$$Q_c = Q_0 + d_1\varepsilon + d_2\varepsilon^2 + d_3\varepsilon^3, \tag{29}$$

where $d_1 = 0.4069$, $d_2 = -0.1669$, $d_3 = 0.075$.

Investigation of the values of v_{opt}, β_{opt} and U_{opt} at given values of Q showed that the above regression equations are in excellent agreement with the numerical data in the range $0.2 \leqslant Q \leqslant 12.0$, which covers most of the practical cases. This is further illustrated in the example given below.

Illustrative example

We now illustrate the usefulness of the results presented, by means of an example problem. Consider a situation in which it is required to determine the optimal dimensions of a circular fin of bore radius 0.05 m, and $Q = 2.0$ needs to be dissipated. The temperature difference between the bore and coolant is 100 K and other variables are, $h = 200\ \mathrm{W \cdot m^{-2} \cdot K^{-1}}$, $k_{cu} = 382\ \mathrm{W \cdot m^{-1} \cdot K^{-1}}$, $k_{al} = 228\ \mathrm{W \cdot m^{-1} \cdot K^{-1}}$. It is required

Table 1. Regression constants for equations (26)–(28) for the constant and optimal fin profile

Profile	a_1	b_1	c_1	a_2	b_2	c_2	a_3	b_3
Constant	0.066	1.081	0.886	0.681	1.429	0.579	1.588	2.448
Variable	0.021	1.005	0.853	1.093	0.895	0.787	0.893	2.584

Table 2. Optimal values for the example from graphical results

Material	Profile	v	U	r_e/r_0	n	w [m]	V [m³]
Aluminum	Constant	0.647	8.41	2.79	—	1.620×10^{-3}	3.456×10^{-4}
	Optimal	0.575	5.29	2.75	2.0	1.055×10^{-3}	2.175×10^{-4}
Copper	Constant	0.647	8.41	2.79	—	2.710×10^{-3}	5.791×10^{-4}
	Optimal	0.575	5.29	2.75	2.0	1.767×10^{-3}	3.643×10^{-4}

Table 3. Optimal values for the example using equations (26)–(28)

Material	Profile	v	U	r_e/r_0	n	w [m]	V [m³]
Aluminum	Constant	0.651	8.660	2.815	—	1.632×10^{-3}	3.550×10^{-4}
	Optimal	0.577	5.353	2.640	2.0	1.159×10^{-3}	2.200×10^{-4}
Copper	Constant	0.651	8.660	2.815	—	2.741×10^{-3}	5.963×10^{-4}
	Optimal	0.577	5.353	2.640	2.0	1.960×10^{-3}	3.686×10^{-4}

to determine the optimal dimensions for a circular fin for the following two cases: (i) constant thickness and (ii) optimal profile. For such a design problem, we can use either the results presented in a graphical form or regression equations. The optimal dimensions of the constant thickness and optimal fin profile are taken from Figs. 3 and 4, and are presented in Table 2. Similar results can also be obtained by using the regression equations (26)–(28) and the results are shown in Table 3. Comparison of the optimal dimensions obtained from graphs and the regression equations compare favorable (within $\pm3\%$). This further consolidates the validity of these equations.

Performance of optimal and constant profile fins

Figures 6 and 7 show the performance for the optimum fin profile ($n = 2$) and the constant thickness aluminum and copper fins. In these figures, we have used $V = 100$ cm³, $k_{cu} = 382$ W·m^{-1}·K^{-1}, $k_{al} = 228$ W·m^{-1}·K^{-1}, $h = 200$ W·m^{-2}·K^{-1}, $r_0 = 0.05$ m, $T_b = 400$ K and $T_\infty = 300$ K. The plot of reduced temperatures (θ) vs reduced radius (ξ) is presented in

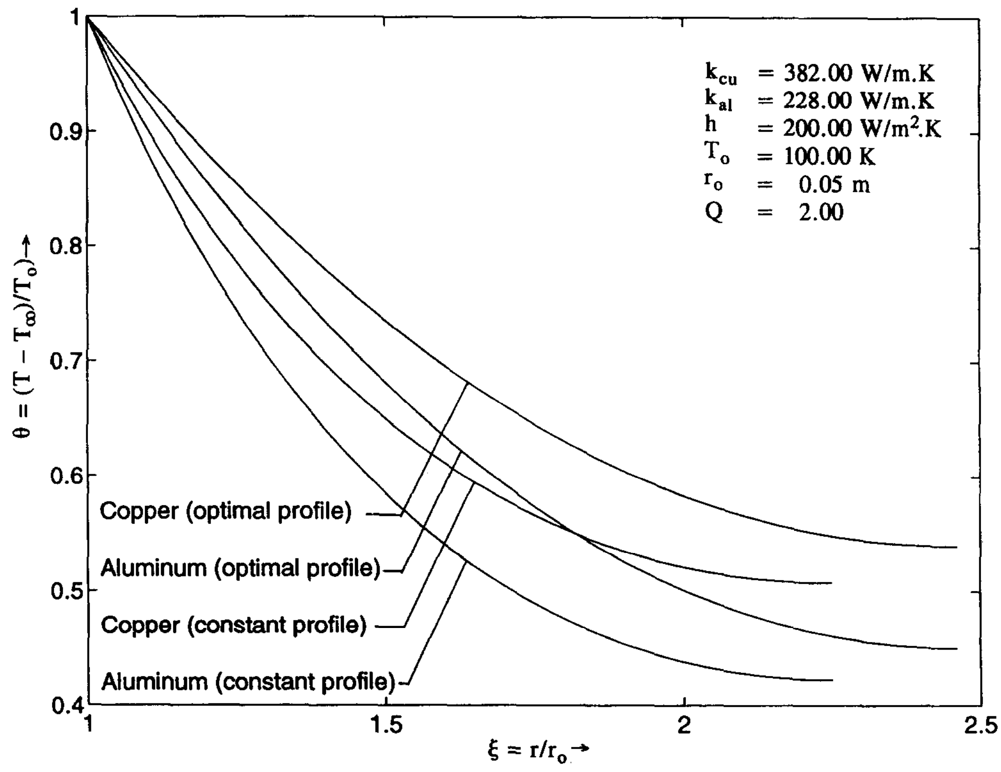

Fig. 6. The reduced temperature (θ) vs reduced radius (ξ) for an optimum profile and constant thickness fins: performance of aluminum and copper fins.

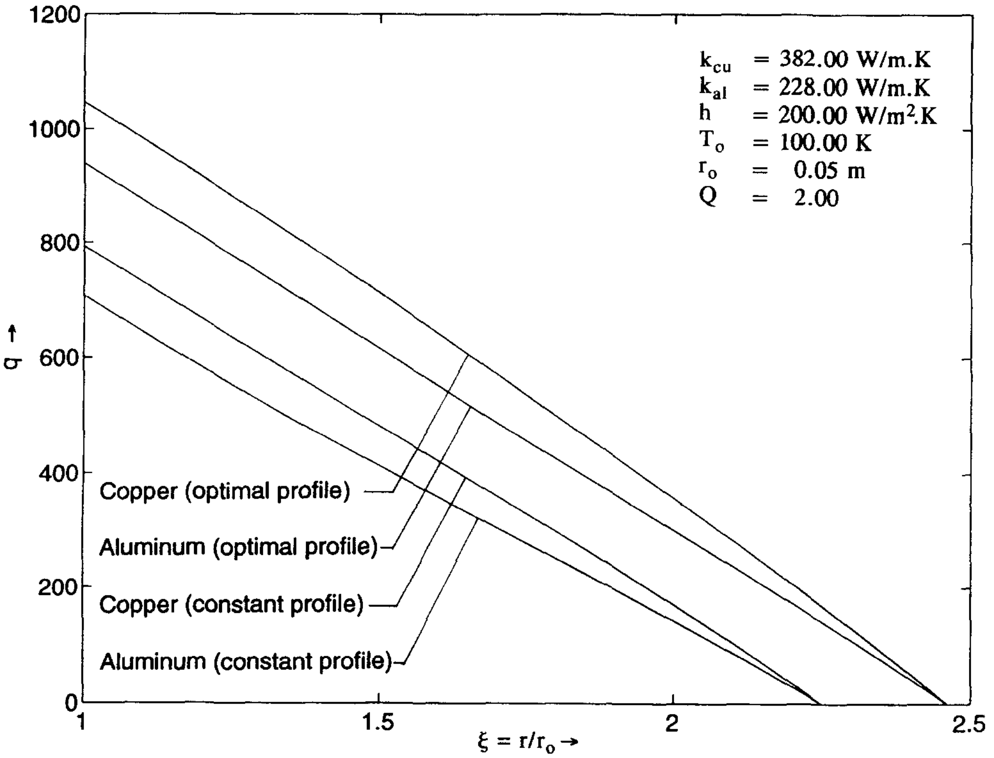

Fig. 7. The heat-transfer rate (q) vs reduced radius (ξ) for an optimum profile and constant thickness fins: performance of aluminum and copper fins.

Fig. 6. As expected, the variable profile copper fin shows the best performance, that is the tip temperature is the highest in this case. This is also confirmed from Fig. 7, where the local heat-transfer rate (q) for any given value of ξ has the highest value.

CONCLUDING REMARKS

The optimum dimensions of circular fins with variable profile have been obtained assuming one-dimensional conduction, and neglecting the effect of curvature and heat-transfer from the tip. The case considered is that of a power law profile with temperature-dependent thermal conductivity. The results are expressed in terms of suitable dimensionless parameters and are presented both in graphical form and in terms of regression equations obtained by standard statistical techniques. These results can be used for design purposes for the range of data we have used in developing the regression equations. It should be noted that the program developed is general, and could take into account the effect of different variations of thermal conductivity and fin profiles suitable for a given application.

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APPENDIX

Formulation of a nonlinear optimization problem

A wide range of engineering problems can be formulated as a general nonlinear optimization problem. The statement of such a problem can be written as [22]

$$\text{Maximize}\, f(x) \quad x \in \mathscr{R}^N \tag{A1}$$

$$\text{Subject to}\ h_k(x) = 0 \quad k = 1, \ldots, K \tag{A2}$$

$$g_j(x) \leqslant 0 \quad j = 1, \ldots, J \tag{A3}$$

$$x_i^{(L)} \leqslant x_i \leqslant x_i^{(U)} \quad i = 1, \ldots, N \tag{A4}$$

where, $h_k(x)$ are equality constraints, $g_j(x)$ are inequality constraints and $x_i^{(L)}$, $x_i^{(U)}$ are bounds on variables $\{x\}$.

The fin problem discussed in this paper fits the definition of the above nonlinear optimization problems. The objective here is to maximize the reduced heat-transfer Q, subject to the constraint that the fin effectiveness E_f is greater than 1. The optimization variables are β and n.